Detailed examples of lift-and-project separation

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In this document we give a detailed description of the separation of lift-and-project for two problems: stein9 and air04.

1 Two cuts of stein9

In the following we give a detailed example of the separation of two cuts for stein9. The problem objective function $\sum x_i$ has been replaced with $\sum i x_i$.

The optimal solution of the LP relaxation is $\bar{x} = (1, \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0)$. We detail the procedure for generating the optimal cuts obtained from $x_2$ and $x_5$.

1.1 Notations

Through this write-up the variables are numbered using the same convention as Balas Perregaard 2003. Namely

- the structural variables of the problem are $x_1, \ldots, x_9$,
- the surplus variables for the constraints are $s_1, \ldots, s_{13}$,
- the surplus variables for the upper bounds are $s_{14}, \ldots, s_{22}$, and
- the surplus for the lower bounds (which are equal to the structural) are $s_{23}, \ldots, s_{31}$.

The problem formulation and all tableaus are given in appendix.

1.2 Cut generated from $x_3$

1.2.1 Initial basis

In the (LP) tableau The row corresponding to $x_3$ is given by:

$$x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 + \frac{2}{3}s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{13} - \frac{2}{3}s_{14} + \frac{2}{3}s_{30} - \frac{1}{3}s_{31} = \frac{2}{3} \quad (1)$$

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The corresponding intersection cut is
\[
\frac{1}{9}s_4 + \frac{1}{9}s_5 + \frac{1}{9}s_6 + \frac{2}{9}s_8 + \frac{2}{9}s_9 + \frac{4}{9}s_{13} + \frac{4}{9}s_{14} + \frac{2}{9}s_{30} + \frac{2}{9}s_{31} \geq \frac{2}{9}
\] (2)

In (CGLP)\textsubscript{3} To compute the objective value of the corresponding solution of (CGLP)\textsubscript{3} this cut has to be normalized to satisfy the normalization constraint \(\sum u + \sum v + u_0 + v_0 = 1\). The normalization factor is \(\frac{3}{16}\) and the resulting cut has violation \(\frac{2}{3}\frac{3}{16} = -0.041667\).

The partition \((M_1, M_2)\) is given by:
\[
M_1 = \{s_9, s_{13}, s_{14}, s_{31}\} \\
M_2 = \{s_4, s_5, s_6, s_8, s_{30}\}
\]
and the corresponding basis is therefore:
\[
\{u_9, u_{13}, u_{14}, u_{31}, v_4, v_5, v_6, v_8, v_{30}\}
\]
(We remind the reader that the basis of (CGLP) is made of the variables \(\alpha, u_0, v_0, u_i \forall i \in M_1, v_i \forall i \in M_2\).)

Note that there is no non-basic variable with a zero coefficient in (1) therefore the basis of (CGLP)\textsubscript{3} corresponding to this basis of (LP) is unique.

1.2.2 First pivot

In the (LP) tableau A reduced cost of \(-\frac{1}{4}\) is found for the variable \(u_{12}\) (corresponding to doing a negative combination with the row in which \(s_{12}\) is basic). Therefore \(s_{12}\) is selected for leaving the basis.

To choose the entering variable, we compute the function \(f_{12}(\gamma)\) which gives the violation of the intersection cut generated from the combination of the source row (here the row where \(x_3\) is basic) and \(\gamma\) times the row in which \(s_{12}\) is basic:
\[
s_{12} + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 - 1s_{13} - 1s_{14} + 1s_{30} - 2s_{31} = 0
\]

The function \(f_{12}(\gamma)\) is plotted in Figure ??.

Each breakpoint in \(f_{12}(\gamma)\) corresponds to a valid pivot in (LP) and the value of the function gives the objective value in the corresponding basis of (CGLP)\textsubscript{3}.

As can be seen from Figure ??, \(f_{12}(\gamma)\) has three breakpoints:
\[
\begin{align*}
\gamma &= -\frac{1}{6} \text{ (with } f_{12}(-\frac{1}{6}) = -0.051282), \\
\gamma &= -\frac{1}{3} \text{ (with } f_{12}(-\frac{1}{3}) = -0.055556) \text{ and} \\
\gamma &= -\frac{2}{3} \text{ (with } f_{12}(-\frac{2}{3}) = -0.05556).
\end{align*}
\]
(note that all the valid pivots combining the two rows are negative). A minimum is attained for \(\gamma = -\frac{2}{3}\) which corresponds to pivoting \(s_{14}\) in the basis of (LP).
Basis of \((CGLP)_3\) after first pivot. After performing the first pivot in \((LP)\), the basis of \((CGLP)_3\) is changed as follows

- \(u_{31}\) leaves the basis and \(v_{31}\) enters,
- \(v_6\) leaves and \(u_6\) enters,
- \(u_{14}\) leaves and \(u_{12}\) enters.

This gives a block pivot which could be performed in \((CGLP)_3\). This block pivot can be explained from the function \(f_{12}(\gamma)\).

Note that the coefficients of the variables in the source row change monotonically as \(\gamma\) decreases. Each breakpoint corresponds to the value of \(\gamma\) at which a coefficient for a particular variable is zero. Let us consider a non-basic variable \(s_i\) which corresponds to a breakpoint for \(\gamma = \gamma_s\) (has a zero in the source row if we take the combination of the source row with \(\gamma_i\) times the row of \(s_1\)) and let us assume that initially \(s_i\) is in \(M_1\). For \(\gamma > \gamma_i\) the variable stays in \(M_1\), for \(\gamma = \gamma_i\) the variable enters the basis and if \(\gamma < \gamma_i\) the variable goes from \(M_1\) to \(M_2\).

Therefore, we can deduce the block pivot by looking at the breakpoints of the function by decreasing values (from right to left) until the minimum value of \(f_{12}(\gamma)\) chosen is attained. The first breakpoint corresponds to \(s_{31}\), the second to \(s_6\) and the third (the one chosen) to \(s_{14}\).

This gives the corresponding block pivot in \((CGLP)_3\) but what is the sequence of pivots which should be performed if we want each basis of \((CGLP)_3\)
visited to be primal feasible?

The answer can again be deduced from $f_{12}(\gamma)$. The first pivot is to get to the first breakpoint of $f_{12}$, $u_{12}$ enters the basis $M_1$ and $u_{31}$ exists. When $\gamma$ continues to decrease the coefficient of $s_{31}$ in the source row becomes positive and $v_{31}$ enters the basis of CGLP. The second breakpoint corresponds to $v_{31}$ entering and $v_6$ exiting. Finally, in the last pivot $u_6$ enters and $u_{14}$ leaves the basis.

These three pivots are summarized in Table 1, it can be verified that each basis of $(CGLP)_3$ encountered during this sequence of pivots is primal feasible.

<table>
<thead>
<tr>
<th>Pivot#</th>
<th>entering</th>
<th>leaving</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$u_{12}$</td>
<td>$u_{31}$</td>
</tr>
<tr>
<td>2</td>
<td>$v_{31}$</td>
<td>$v_6$</td>
</tr>
<tr>
<td>3</td>
<td>$u_6$</td>
<td>$u_{14}$</td>
</tr>
</tbody>
</table>

Table 1: List of pivots in $(CGLP)_3$ corresponding to first pivot in $(LP)$

Is it possible to pivot to the same basis of $(CGLP)_3$ by a sequence of degenerate pivots  By analysing this example we can answer negatively to the following question:

**Open Question 1** Are the nonadjacent bases of CGLP that correspond to the two adjacent bases of LP, associated with adjacent solutions of the CGLP? In other words, is the solution of CGLP associated with J’ obtainable from that associated with J through a sequence of pivots (in the CGLP) of which only the last one is nondegenerate?

We will argue that in this pivot for the stein9 example there is no such sequence of degenerate pivots followed by one last non-degenerate pivot. First we argue that the initial basis of (CGLP) is non-degenerate and therefore the first pivot is non-degenerate. Then we argue that there is no valid pivot from the initial basis to a basis with a violation similar to the one obtained from the cut in the adjacent basis of (LP).

The starting basis of $(CGLP)_3$ is non degenerate since all the basic variables have a non-zero value. This can be verified numerically or using the proof of Theorem 4A of [?] page 18 showing the equivalence between lift-and-project cuts and intersection cuts. Formula (18) in the proof of Theorem 4A gives the values of the variables in the basis of $(CGLP)_3$ corresponding to a given basis of $(LP)$:

$$
\theta u_j = \pi^2_j - \pi^1_j, j \in M_1,
\theta v_j = \pi^1_j - \pi^2_j, j \in M_2,
\theta u_0 = 1 - \pi_{k0},
\theta v_0 = \pi_{k0}
$$
where \( \theta > 0 \) is a constant corresponding to the normalization constraint, \( \pi^1_j = \overline{\sigma}_{kj}(1 - \overline{\sigma}_{k0}) \) and \( \pi^2_j = -\overline{\sigma}_{kj} \overline{\sigma}_{k0} \).

From this formula it can be seen that if all the coefficient for the non-basic variables in (LP) are non-zero the corresponding solution of (CGLP) has no basic variable with a value of zero and is non-degenerate.

Since there is no degenerate pivot in the initial basis if the answer to the question was true there should be a single pivot which lead to a cut with violation -0.055556 (the violation of the cut obtained as a result of the pivot in (LP)). This pivot could eventually be followed by degenerate pivots. But it appears that there is no pivot to an adjacent (primal feasible) basis in CGLP which gives a cut with violation \(-0.055556\). This I, verified numerically with the code. There is actually only one variable with a negative reduced cost (the one we have chosen) and the only adjacent primal feasible basis is with \(s_{12}\) entering in \(M_1\) and \(s_{31}\) exiting. The resulting basis of (CGLP) has an objective value of \(-0.051282\) and therefore at least one more non-degenerate pivot is necessary.

1.2.3 Second pivot

In the (LP) tableau After the first pivot, the source row reads

\[
x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} + 0s_{30} + 1s_{31} = \frac{2}{3}
\]

(3)

The corresponding solution of (CGLP)_3 has objective value \(-0.05556\).

A negative reduced cost is found for the variable corresponding to the row basic in variable \(s_7\). Minimizing \(f_7(\gamma)\) for this row we find that the best cut is obtained if \(s_{31}\) enters the basis.

The next basis is optimal for (CGLP)_3. The intersection cut has violation \(-0.066667\).

Second basis of CGLP Note that in the new source row of (LP) some non-basic variables have a zero coefficient. Therefore the basis of (CGLP)_3 is not uniquely defined anymore. When choosing \(s_7\) to enter the basis, pivots are implicitly made in (CGLP)_3:

- the variable \(v_8\) is replaced by \(u_8\),
- the variable \(u_{13}\) is replaced by \(v_{13}\),
- the variable \(v_{30}\) is replaced by \(u_{30}\).

The resulting basis is given by:

\[\{u_6, u_8, u_9, u_{12}, u_{30}, v_4, v_5, v_{13}, v_{31}\}\]

Second block pivot To minimize \(f_7(\gamma)\) only one discontinuity point is encountered which corresponds to \(v_7\) entering the basis \(v_{31}\) leaving it.
Table 2: List of pivots in \((CGLP)_6\)

<table>
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<tr>
<th>Pivot#</th>
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<tbody>
<tr>
<td>1</td>
<td>(u_{12})</td>
<td>(v_{30})</td>
</tr>
<tr>
<td>2</td>
<td>(u_{30})</td>
<td>(u_{31})</td>
</tr>
<tr>
<td>3</td>
<td>(v_{31})</td>
<td>(u_{14})</td>
</tr>
</tbody>
</table>

1.2.4 Summary

In total 2 pivots are performed in (LP) to obtain the optimal cut. These two pivots correspond to 7 pivots in \((CGLP)_3\) of which only 3 are degenerate. The final source row is:

\[
\begin{align*}
    x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + 0s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 - \frac{1}{3}s_9 - \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 0s_{30} &= \frac{2}{3}, \\
    \frac{1}{9}s_4 + \frac{1}{9}s_5 + \frac{1}{9}s_7 + \frac{1}{9}s_8 + \frac{2}{9}s_9 + \frac{2}{9}s_{12} + \frac{2}{9}s_{13} &\geq \frac{2}{9}
\end{align*}
\]

The corresponding intersection cut is:

\[
\begin{align*}
    x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + 0s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 - \frac{1}{3}s_9 - \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 0s_{30} &= \frac{2}{3}, \\
    \frac{1}{9}s_4 + \frac{1}{9}s_5 + \frac{1}{9}s_7 + \frac{1}{9}s_8 + \frac{2}{9}s_9 + \frac{2}{9}s_{12} + \frac{2}{9}s_{13} &\geq \frac{2}{9}
\end{align*}
\]

The cut violation is \(\frac{2}{9}\) and the normalization factor \(\frac{10}{3}\), which give an optimal value of \(\frac{1}{15} = 0.066667\) for \((CGLP)_3\).

1.3 Cut generated on \(x_6\)

The optimal cut from \(x_6\) is obtained with only one pivot in the LP tableau. The source row is

\[
\begin{align*}
    x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{2}{3}s_6 + \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} + \frac{1}{3}s_{30} - \frac{2}{3}s_{31} = \frac{1}{3}
\end{align*}
\]

The objective value at the basis of \((CGLP)_6\) corresponding to the optimal basis of (LP) 0.047619. A negative reduced cost for \((CGLP)_6\) is found for the variable corresponding to the row in which \(s_{12}\) is basic.

Minimizing \(f_{12}(\gamma)\) the following pivots are performed
- \(v_{30}\) leaves the basis and \(u_{30}\) enters,
- \(u_{31}\) leaves and \(v_{31}\) enters,
- \(u_{14}\) exits the basis and \(u_{12}\) enters.

The resulting cut has violation \(-0.083333\) and is optimal for \((CGLP)_6\).

2 One cut of air04

As seen on the two cuts derived for stein9 one pivots in (LP) can correspond to several non-degenerate pivots in (CGLP). In the following we study (in less
details) the separation of one cut for a large size problem: air04. The problem has 8904 variables (all binaries) and 823 constraints.

We study the separation of the lift-and-project cut corresponding to variable $x_{895}$ for the initial linear relaxation (the cut is separated in the reduced space).

The violation of the lift-and-project cut corresponding to the initial tableau is $-0.000091$, 20 pivots are performed in (LP) to obtain a cut with violation $-0.001317$ (the procedure stops because the limit of 20 pivots is attained).

The function $f(\gamma)$ for the first pivot is plotted in Figure ???. The function has 579 discontinuity points each corresponding to a pivot. Its minimum is attained for $\gamma = 0.706161$. This pivot in (LP) corresponds to 194 pivots in (CGLP).

Table ?? gives the number of pivots in (CGLP) corresponding to the 20 pivots performed in (LP). In total the 20 pivots in (LP) correspond to 1397 pivots in (CGLP).

References

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<th>Pivot #</th>
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<tr>
<td>20</td>
<td>-0.001317</td>
<td>44</td>
</tr>
</tbody>
</table>

Table 3: The 20 pivots performed to obtain the lift-and-project of air04

A Stein9 formulation

\[
\begin{align*}
\min & \sum_{i=1}^{9} x_i \\
\text{Subject To} & \\
& x_2 + x_3 + x_4 \geq 1 \\
& x_1 + x_3 + x_5 \geq 1 \\
& x_1 + x_2 + x_6 \geq 1 \\
& x_5 + x_6 + x_7 \geq 1 \\
& x_4 + x_6 + x_8 \geq 1 \\
& x_4 + x_5 + x_9 \geq 1 \\
& x_1 + x_8 + x_9 \geq 1 \\
& x_2 + x_7 + x_9 \geq 1 \\
& x_3 + x_7 + x_8 \geq 1 \\
& x_1 + x_4 + x_7 \geq 1 \\
& x_2 + x_5 + x_8 \geq 1 \\
& x_3 + x_6 + x_9 \geq 1 \\
\sum_{i=1}^{9} x_i & \geq 4 \\
x & \in \{0,1\}^9
\end{align*}
\]
## B Successive tableaus

### B.1 Optimal tableau

The optimal basis is

\[ B = \{ s_1, s_2, s_3, x_7, x_5, x_4, s_7, x_2, x_3, s_{10}, s_{11}, s_{12}, x_6 \} \]

(this is not totally accurate, actually all \( x \) variables are basic by hypothesis in BP03, the basic variables are the surplus to lower bounds corresponding to these.)

The optimal tableau is (the first variable is always the basic variable in the row).

\[
\begin{align*}
    s_1 + 1s_4 + 0s_5 + 0s_6 + 0s_8 + 0s_9 - 1s_{13} - 1s_{14} + 1s_{30} + 1s_{31} &= 1 \\
    s_2 + 0s_4 + 1s_5 + 0s_6 + 1s_8 + 0s_9 - 1s_{13} + 0s_{14} + 0s_{30} + 0s_{31} &= 1 \\
    s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_8 + 1s_9 - 1s_{13} + 0s_{14} + 0s_{30} + 0s_{31} &= 1 \\
    x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 - \frac{1}{3}s_6 - \frac{2}{3}s_8 - \frac{2}{3}s_9 + \frac{2}{3}s_{13} + \frac{2}{3}s_{14} + \frac{1}{3}s_{30} + \frac{1}{3}s_{31} &= \frac{1}{3} \\
    x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{1}{3}s_6 + \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} - \frac{2}{3}s_{30} + \frac{1}{3}s_{31} &= \frac{1}{3} \\
    x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{2}{3}s_6 - \frac{1}{3}s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{13} + \frac{1}{3}s_{14} + \frac{2}{3}s_{30} + \frac{2}{3}s_{31} &= \frac{2}{3} \\
    s_7 + 0s_4 + 0s_5 + 0s_6 + 0s_8 + 0s_9 + 0s_{13} + 1s_{14} - 1s_{30} - 1s_{31} &= -0 \\
    x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 - \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} + \frac{2}{3}s_{30} + \frac{1}{3}s_{31} &= \frac{2}{3} \\
    x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 + \frac{1}{3}s_8 - \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} + \frac{2}{3}s_{30} - \frac{1}{3}s_{31} &= \frac{1}{3} \\
    x_{10} + 0s_4 - 1s_5 - 1s_6 - 1s_8 - 1s_9 + 1s_{13} + 2s_{14} + 1s_{30} + 1s_{31} &= 1 \\
    s_{11} + 0s_4 + 1s_5 + 0s_6 + 0s_8 + 1s_9 - 1s_{13} - 1s_{14} - 2s_{30} + 1s_{31} &= -0 \\
    s_{12} + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 - 1s_{13} - 1s_{14} + 1s_{30} - 2s_{31} &= -0 \\
    x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{2}{3}s_6 + \frac{1}{3}s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{13} - \frac{1}{3}s_{14} + \frac{1}{3}s_{30} + \frac{2}{3}s_{31} - \frac{2}{3}s_{31} &= \frac{1}{3} 
\end{align*}
\]
B.2 After the first pivot for \((CGLP)_3\)

\[
\begin{align*}
s_1 + 1s_4 + 0s_5 - 1s_6 - 1s_8 + 0s_9 - 1s_{12} + 0s_{13} + 0s_{30} + 3s_{31} &= 1 \\
s_2 + 0s_4 + 1s_5 + 0s_6 + 1s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} &= 1 \\
s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} &= 1 \\
x_7 - \frac{1}{3} s_4 - \frac{1}{3} s_5 + \frac{1}{3} s_6 + 0s_8 - \frac{2}{3} s_9 + \frac{2}{3} s_{12} + 0s_{13} + 1s_{30} - 1s_{31} &= \frac{1}{3} \\
x_5 - \frac{1}{3} s_4 + \frac{2}{3} s_5 - \frac{2}{3} s_6 + 0s_8 + \frac{1}{3} s_9 - \frac{1}{3} s_{12} + 0s_{13} - 1s_{30} + 1s_{31} &= 1 \\
x_4 + \frac{1}{3} s_4 - \frac{2}{3} s_5 - \frac{1}{3} s_6 + 0s_8 - \frac{1}{3} s_9 + \frac{1}{3} s_{12} + 0s_{13} + 1s_{30} + 0s_{31} &= \frac{2}{3} \\
s_7 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 0s_{30} - 3s_{31} &= -0 \\
x_2 + \frac{1}{3} s_4 + \frac{1}{3} s_5 - \frac{1}{3} s_6 - 1s_8 + \frac{2}{3} s_9 - \frac{2}{3} s_{12} + 0s_{13} - 1s_{30} + 2s_{31} &= \frac{2}{3} \\
x_3 + \frac{1}{3} s_4 + \frac{1}{3} s_5 - \frac{1}{3} s_6 + 0s_8 - \frac{1}{3} s_9 + \frac{2}{3} s_{12} + 0s_{13} + 0s_{30} + 1s_{31} &= \frac{2}{3} \\
s_{10} + 0s_4 - 1s_5 + 1s_6 + 1s_8 - 1s_9 + 2s_{12} - 1s_{13} + 3s_{30} - 3s_{31} &= 1 \\
s_{11} + 0s_4 + 1s_5 - 1s_6 - 1s_8 + 1s_9 - 1s_{12} + 0s_{13} - 3s_{30} + 3s_{31} &= 0 \\
x_1 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 1s_{30} - 2s_{31} &= 1 \\
x_6 - \frac{1}{3} s_4 - \frac{1}{3} s_5 + \frac{1}{3} s_6 + 0s_8 + \frac{1}{3} s_9 - \frac{1}{3} s_{12} + 0s_{13} + 0s_{30} + 0s_{31} &= \frac{1}{3}
\end{align*}
\]
B.3 After second pivot of \((CGLP)_3\)

\[
\begin{align*}
 s_1 + 1s_4 + 0s_5 + 0s_6 + 1s_7 + 0s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} &= 1 \\
 s_2 + 0s_4 + 1s_5 + 0s_6 + 0s_7 + 1s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} &= 1 \\
 s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_7 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} + 0s_{30} &= 1 \\
 x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + 0s_6 - \frac{1}{3}s_7 - \frac{1}{3}s_8 - \frac{2}{3}s_9 + \frac{1}{3}s_{12} + \frac{1}{3}s_{13} + 1s_{30} &= \frac{1}{3} \\
 x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{1}{3}s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 + \frac{1}{3}s_9 + 0s_{12} - \frac{1}{3}s_{13} - 1s_{30} &= \frac{1}{3} \\
 x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{1}{3}s_6 + 0s_7 + 0s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{12} + 0s_{13} + 1s_{30} &= \frac{2}{3} \\
 x_9 + 0s_4 + 0s_5 - \frac{1}{3}s_6 - \frac{1}{3}s_7 - \frac{1}{3}s_8 + 0s_9 - \frac{1}{3}s_{12} + \frac{1}{3}s_{13} + 0s_{30} &= 0 \\
 x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + \frac{1}{3}s_6 + \frac{2}{3}s_7 - \frac{1}{3}s_8 + \frac{2}{3}s_9 + 0s_{12} - \frac{2}{3}s_{13} - 1s_{30} &= \frac{2}{3} \\
 x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 + 0s_6 + \frac{1}{3}s_7 + \frac{1}{3}s_8 - \frac{1}{3}s_9 - \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 0s_{30} &= \frac{2}{3} \\
 s_{10} + 0s_4 - 1s_5 + 0s_6 - 1s_7 + 0s_8 - 1s_9 + 1s_{12} + 0s_{13} + 3s_{30} &= 1 \\
 s_{11} + 0s_4 + 1s_5 + 0s_6 + 1s_7 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} - 3s_{30} &= 0 \\
 x_1 + 0s_4 + 0s_5 + \frac{1}{3}s_6 - \frac{2}{3}s_7 + \frac{1}{3}s_8 + 0s_9 + \frac{1}{3}s_{12} - \frac{1}{3}s_{13} + 1s_{30} &= 1 \\
 x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_7 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} + 0s_{30} &= \frac{1}{3}
\end{align*}
\]
B.4 After first pivot of $(CGLP)_6$

\[ s_1 + 1s_4 + 0s_5 - 1s_6 - 1s_8 + 0s_9 - 1s_{12} + 0s_{13} + 0s_{30} + 3s_{31} = 1 \]
\[ s_2 + 0s_4 + 1s_5 + 0s_6 + 1s_8 + 0s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} = 1 \]
\[ s_3 + 0s_4 + 0s_5 + 1s_6 + 0s_8 + 1s_9 + 0s_{12} - 1s_{13} + 0s_{30} + 0s_{31} = 1 \]
\[ x_7 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_8 - \frac{2}{3}s_9 + \frac{2}{3}s_{12} + 0s_{13} + 1s_{30} - 1s_{31} = \frac{1}{3} \]
\[ x_5 - \frac{1}{3}s_4 + \frac{2}{3}s_5 - \frac{2}{3}s_6 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} - 1s_{30} + 1s_{31} = \frac{1}{3} \]
\[ x_4 + \frac{1}{3}s_4 - \frac{2}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 + \frac{1}{3}s_{12} + 0s_{13} + 1s_{30} + 0s_{31} = \frac{2}{3} \]
\[ x_7 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 0s_{30} - 3s_{31} = -0 \]
\[ x_2 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 - 1s_8 + \frac{2}{3}s_9 - \frac{2}{3}s_{12} + 0s_{13} - 1s_{30} + 2s_{31} = \frac{2}{3} \]
\[ x_3 + \frac{1}{3}s_4 + \frac{1}{3}s_5 - \frac{1}{3}s_6 + 0s_8 - \frac{1}{3}s_9 + \frac{2}{3}s_{12} + 0s_{13} + 0s_{30} + 1s_{31} = \frac{2}{3} \]
\[ s_{10} + 0s_4 - 1s_5 + 1s_6 + 1s_8 - 1s_9 + 2s_{12} - 1s_{13} + 3s_{30} - 3s_{31} = 1 \]
\[ s_{11} + 0s_4 + 1s_5 - 1s_6 - 1s_8 + 1s_9 - 1s_{12} + 0s_{13} - 3s_{30} + 3s_{31} = 0 \]
\[ x_1 + 0s_4 + 0s_5 + 1s_6 + 1s_8 + 0s_9 + 1s_{12} - 1s_{13} + 1s_{30} - 2s_{31} = 1 \]
\[ x_6 - \frac{1}{3}s_4 - \frac{1}{3}s_5 + \frac{1}{3}s_6 + 0s_8 + \frac{1}{3}s_9 - \frac{1}{3}s_{12} + 0s_{13} + 0s_{30} + 0s_{31} = \frac{1}{3} \]